In the Name of GOD

* Signal Space

Similar to what we said in Vector Space, Some Concepts Can be explained in signal space too.

- Inner product of two Signals

The inner product of any two Signal $x_1(t)$ & $x_2(t)$ is defined as below

$$\langle x_1(t), x_2(t) \rangle = \int_a^a x_1(t)\, x_2^*(t)\, dt$$

in this part we assume that our signal are defined in the range $t \in [a, b]$

- Orthogonality of two signals

If the inner product of any two signals $x_1(t)$ & $x_2(t)$ is equal to zero, then these two signals are orthogonal, i, e,

$$\langle x_1(t), x_2(t) \rangle = 0 \iff x_1(t) \perp x_2(t)$$

– Signal Norm

For any signal $x(t)$, the signal norm is defined by the use of inner product Concept, i.e,

$$\underbrace{\|x(t)\|_2}_{\text{The Norm of } x(t)} = \langle x(t), x(t) \rangle^{1/2} = \left[\int_a^b x(t)\, x^*(t)\, dt \right]^{1/2}$$

$$= \left[\underbrace{\int_a^b |x(t)|^2\, dt}_{\text{Energy of } x(t) = E} \right]^{1/2} = \sqrt{E} = \|x(t)\|_2$$

- Set of orthonormal signals

The set of signal $\{x_i(t)\}_{i=1}^{m}$ is called an orthonormal set of signals, if these signal are orthogonal to each other and have uint norm, i.e,

$$\forall\ i \neq j\ ; \quad \langle x_i(t), x_j(t) \rangle = 0$$

&

$$\forall\ i\ ; \quad \| x_i(t) \| = 1$$

or equivalently

$$\langle x_i(t), x_j(t) \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

- Linear Independency of signals

The set of signals $\{x_i(t)\}_{i=1}^{m}$ is a set of linear independent signals, if no signal in this set can be represented as the linear combination of the other signals. So, if the eauation

$$a_1 x_1(t) + a_2 x_2(t) + \dots + a_m x_m(t) = 0$$

just have the zero answer $a_1 = a_2 = \dots = a_m = 0$

we con conclude that $\{x_i(t)\}_{i=1}^{m}$ are linearly independet.

otherwise $\{x_i(t)\}_{i=1}^{m}$ are linear dependent.

- Triangle inequality

In signal space, for any signals $x_1(t)$ & $x_2(t)$ we have the Triangle inequality as below

$$\|x_1(t)+x_2(t)\| \leqslant \|x_1(t)\| + \|x_2(t)\|$$

equality happens when $x_1(t)$ & $x_2(t)$ are in the same direction, mean

$$\exists a \in \mathbb{R}^+ \;; \quad x_1(t) = a\, x_2(t)$$

Prove this inequality as an Exeercise.

$$\| x_1(t) + x_2(t) \|^2 = \| x_1(t) \|^2 + \| x_2(t) \|^2 + 2\,\mathrm{Re}\{ \langle x_1(t), x_2(t) \rangle \}$$

$$\| \underline{v}_1 + \underline{v}_2 \|^2 = \| \underline{v}_1 \|^2 + \| \underline{v}_2 \|^2 + 2\,\mathrm{Re}\{ \langle \underline{v}_1, \underline{v}_2 \rangle \}$$

- Cauchy - Schwartz in equality

For any signals $x_1(t)$ & $x_2(t)$ in the signal space, Cauchy - schwartz inequality is as follows,

$$| \langle x_1(t), x_2(t) \rangle | \leq \| x_1(t) \| \, \| x_2(t) \|$$

$$\underbrace{\left| \int_a^b x_1(t)\, x_2^*(t)\, dt \right| \leq \left[\int_a^b |x_1(t)|^2\, dt \right]^{1/2} \left[\int_a^b |x_2(t)|^2\, dt \right]^{1/2}}_{\text{The general rep of Caushy-shwartz inequality}}$$

$$\left| \langle x_1(t), x_2(t) \rangle \right| \leq \underbrace{\langle x_1(t), x_1(t) \rangle^{1/2}}_{\|x_1(t)\|} \underbrace{\langle x_2(t), x_2(t) \rangle^{1/2}}_{\|x_2(t)\|}$$

equality happens when we have

$$\exists\, a \in \mathbb{C} \quad ; \quad x_1(t) = a\, x_2(t)$$

In the signal space we can find a set of orthonormal basis functions $\{\varphi_i(t)\}_{i=1}^{N}$ for the signal space spanned by the set of signals $\{s_i(t)\}_{i=1}^{M}$.

by the use of Gram-Schemit algorithm. So, we can present any signal in the signal space by a vector in the corresponding vector space.

But, first it is important for us to explain about the orthogonal expantion of any signal with finite energy.

This expansion help us, represent the signal by a vector in vector space.

- Orthogonal Expansion of signals with finite Energy

In signal space any deterministic signal $s(t)$ that has finite energy, means

$$E_s = \int_a^b |s(t)|^2 \, dt < \infty$$

can be represented by the linear Combination of a set of orthonormal basis function $\{\varphi_i(t)\}_{i=1}^{N}$, in which we have

$$\langle \varphi_i(t), \varphi_j(t) \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

we can represent, $s(t)$, as a linear combination of $\{\varphi_i(t)\}_{i=1}^{N}$ as follows,

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t) = s_1 \varphi_1(t) + s_2 \varphi_2(t) + \cdots + s_N \varphi_N(t)$$

in which $\hat{s}(t)$ is an approximation (estimation) of $s(t)$.

In this part we will show that $\hat{s}(t)$ is a good enugh approximation (estimation) of $s(t)$ so we can write (under some situation) in some sense (MS)

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t) = s(t)$$

in which $\{s_i\}_{i=1}^{N}$ is called the orthogonal expansion coefficients

Suppose that we want to estimate $s(t)$ as the linear Combinatio of $\{\varphi_i(t)\}_{i=1}^{N}$ as below

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$

So, first we introduced the error function $e(t)$ as,

$$e(t) = s(t) - \hat{s}(t)$$

To find a good enough estimation of $s(t)$ in the form of

$\hat{s}(t)$, we should minimize the energy of error function $e(t)$

$$\underbrace{\mathcal{E}_e}_{\text{the energy of } e(t)} = \int_a^b |e(t)|^2 \, dt = \int_a^b |s(t) - \hat{s}(t)|^2 \, dt$$

$$\text{Minmize } \mathcal{E}_e \xrightarrow{\text{find}} \{s_i\}_{i=1}^{N}$$

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$

To minimize E_e with respect to $\{s_i\}_{i=1}^{N}$, we should differentiate E_e with respect to $\{s_i\}_{i=1}^{N}$ and set the first derivatives equal to zero. So we have a set of equation, solving them will get us $\{s_i\}_{i=1}^{N}$

$$\left.\begin{aligned} \frac{\partial}{\partial s_1} E_e &= 0 \\ \frac{\partial}{\partial s_2} E_e &= 0 \\ &\vdots \\ \frac{\partial}{\partial s_N} E_e &= 0 \end{aligned}\right\} \xrightarrow[\text{solving}]{\text{equation}} \int_a^b e(t)\,\hat{\varphi}_i(t)\,dt = 0$$

$$i = 1, \ldots, N$$

$e(t) = s(t) - \hat{s}(t)$

$\Longrightarrow$

$$\int_a^b \left(s(t) - \hat{s}(t) \right) \varphi_i^*(t)\, dt = 0$$

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$

$\Longrightarrow$ we can find $\{ s_i \}_{i=1}^{N}$

Another solution for this problem, come from Estimation theory.
when we want to estimate a signal $s(t)$ in the form of

$\hat{s}(t)$ by the $\underbrace{\text{Minimum Mean Square Error}}_{\text{MMSE}}$ Criteria

we should Minimize $|\underbrace{s(t)-\hat{s}(t)}_{e(t)}|^2$ (deterministic signal)

In MMSE estimation, we have the orthogonality rule

that says, always $e(t)$ is orthogonal to our information

about $s(t)$. In this problem (finding $s(t)$ as a linear

Combination of $\{\varphi_i(t)\}_{i=1}^{N}$) our information is our

basis function $\{\varphi_i(t)\}_{i=1}^{N}$. So, we can write

$\langle e(t), \varphi_i(t) \rangle = 0$

$$\int_a^b e(t)\, \varphi_i^*(t)\, dt = 0 \quad , \; i = 1, \dots, N$$

Any way, solving the above equations, lead us to find $\{S_i\}_{i=1}^N$

$$\int_a^b e(t)\, \varphi_i^*(t)\, dt = \int_a^b (s(t) - \hat{s}(t))\, \varphi_i^*(t)\, dt = 0$$

$$\Longrightarrow \int_a^b \hat{s}(t)\, \varphi_i^{*}(t)\, dt = \int_a^b s(t)\, \varphi_i^{*}(t)\, dt$$

$$\Longrightarrow \int_a^b \left(\sum_{j=1}^{N} s_j \varphi_j(t) \right) \varphi_i^{*}(t)\, dt = \int_a^b s(t)\, \varphi_i^{*}(t)\, dt$$

$$\uparrow$$

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t) \quad , \quad \boxed{\int \varphi_i(t)\, \varphi_j^{*}(t)\, dt = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}}$$

$$\Longrightarrow \int_a^b s_i\, \varphi_i^{*}(t)\, \varphi_i(t)\, dt = \int_a^b s(t)\, \varphi_i^{*}(t)\, dt$$

$$\Rightarrow \quad s_i \underbrace{\int_a^b |\varphi_i(t)|^2 \, dt}_{1} = \int_a^b s(t)\, \varphi_i^*(t)\, dt$$

$$\Rightarrow \quad s_i = \int_a^b s(t)\, \varphi_i^*(t)\, dt = \langle s(t), \varphi_i(t) \rangle$$

$i = 1, 2, \ldots, N$

It means that the Coefficients of orthogonal Expansion of $s(t)$ using the orthonormal basis functions $\{\varphi_i(t)\}_{i=1}^{N}$

is equal to its projection on $\{\varphi_i(t)\}_{i=1}^{N}$

$$s_i = \int_a^b s(t)\,\varphi_i^*(t)\,dt = \langle s(t), \varphi_i(t) \rangle$$

s_i is the projection of $s(t)$ on $\varphi_i(t)$

$\Longrightarrow$ $\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$ is the projection of $s(t)$ on the signal space spanned by $\{\varphi_i(t)\}_{i=1}^{N}$

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$ (minimized ε_e)

in which $$s_i = \int_a^b s(t) \varphi_i^*(t)\, dt = \langle s(t), \varphi_i(t) \rangle$$

for $i = 1, \dots, N$

To find the orthogonal expansion of $s(t)$, we minimized the energy of error function $e(t)$, Named it ε_e. So, in the next part we want to find the minimum energy of $e(t)$, call it $\mathcal{E}_{e_{min}}$

$$\mathcal{E}_{e_{min}} = \int_a^b |e(t)|^2 \, dt \Bigg|_{\left\{ s_i = \langle s(t), \varphi_i(t) \rangle \right\}_{i=1}^{N}}$$

$$\mathcal{E}_{e_{min}} = \int_a^b e(t) \, e^*(t) \, dt$$

$$\left\{ \begin{aligned} e(t) &= s(t) - \hat{s}(t) \\ \hat{s}(t) &= \sum_{i=1}^{N} s_i \varphi_i(t) \\ s_i &= \langle s(t), \varphi_i(t) \rangle \end{aligned} \right\}$$

$$\mathcal{E}_{e_{min}} = \int_a^b e(t) \left(s(t) - \underbrace{\sum_{i=1}^{N} s_i \varphi_i(t)}_{\hat{s}(t)} \right)^* dt$$

$$e(t) \perp \{\varphi_i(t)\}_{i=1}^{N} \Longrightarrow e(t) \perp \hat{s}(t)$$

$$\epsilon_{e_{min}} = \int_a^b e(t)\, s^*(t)\, dt$$

$$\Rightarrow \epsilon_{e_{min}} = \int_a^b \left(s(t) - \sum_{i=1}^{N} s_i \varphi_i(t) \right) s^*(t)\, dt$$

$$\epsilon_{e_{min}} = \underbrace{\int_a^b |s(t)|^2\, dt}_{\epsilon_s} - \int_a^b \sum_{i=1}^{N} s_i \varphi_i(t)\, s^*(t)\, dt$$

$$\epsilon_{e_{min}} = \epsilon_s - \sum_{i=1}^{N} s_i \underbrace{\int_a^b \varphi_i(t)\, s^*(t)\, dt}_{s_i^*} = \epsilon_s - \sum_{i=1}^{N} |s_i|^2$$

$$\Rightarrow \quad \boxed{\mathcal{E}_{e_{min}} = \mathcal{E}_s - \sum_{i=1}^{N} |s_i|^2}$$

$$\mathcal{E}_e = \int_a^b |s(t) - \hat{s}(t)|^2 \, dt$$

$$e(t) = s(t) - \hat{s}(t) \quad , \quad \mathcal{E}_e = \text{Energy of } e(t)$$

$$\hat{s}(t) = \sum_{i=1}^{N} s_i \, \varphi_i(t) \xrightarrow[\mathcal{E}_e]{\text{minimize}} \left\{ s_i = \langle s(t), \varphi_i(t) \rangle \right\}_{i=1}^{N}$$

If $\mathcal{E}_{e_{min}}$ is equal to zero we can conclude that

$\hat{s}(t)$ is equal to $s(t)$ in mean square sense.

$$s(t) \overset{MS}{=\!=} \hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$

As you may know, the equality in MS sense is a powerful mathematical equality. So, we can conclude that $\hat{s}(t)$ is a good enough approximation (estimation) of $s(t)$. Therefore we can represent $s(t)$ as an orthogonal expansion as $\hat{s}(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$

On the other hand, if the set of $\{\varphi_i(t)\}_{i=1}^{N}$ is a Complete orthonormal set, alway $\epsilon_{e_{min}}$ is equal to zero and we can conclude that

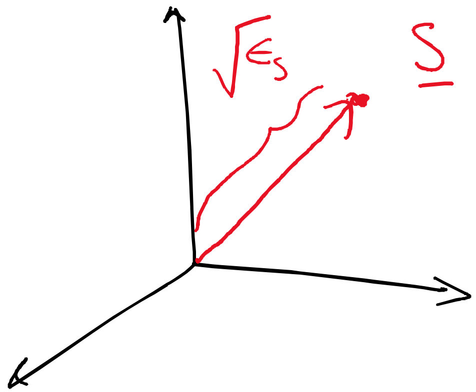
$$s(t) \overset{MS}{=\!=} \sum_{i=1}^{N} s_i \varphi_i(t) = \hat{s}(t)$$

$$S_i = \langle s(t), \varphi_i(t) \rangle , \quad i = 1, \dots, N$$

$$E_s = \sum_{i=1}^{N} |s_i|^2$$

So, under that situation, we can show any signal $s(t)$ in the signal space, by a vector $\underline{s} = (s_1, s_2, \dots, s_N)^T$ in corresponding vector space

$$s(t) = \sum_{i=1}^{N} s_i \varphi_i(t)$$



$$\xrightarrow[\text{space}]{\text{vector}} \quad \underline{s} = (s_1, s_2, \dots, s_N)^T$$

$$\left\{ s_i = \langle s(t), \varphi_i(t) \rangle \right\}_{i=1}^{N}, \quad E_s = \sum_{i=1}^{N} |s_i|^2 = \|\underline{s}\|^2$$